2.1 Basic Concepts of Propositional Logic

2.2 Equivalence Calculus of Propositional Logic

2.3 Normal Forms

- **2.3.1 Disjunctive Normal Form (DNF) and Conjunctive Normal Form (CNF)**
  - Simple Disjunctive Form and Simple Conjunctive Form
  - Disjunctive Normal Form and Conjunctive Normal Form
- **2.3.2 Principal Disjunctive Normal Form (PDNF) and Principal Conjunctive Normal Form (PCNF)**
  - Minterms and Maxterms
  - Principal Disjunctive Normal Form and Principal Conjunctive Normal Form
  - Applications of Principal Normal Form

- **Literal**: A general term for a propositional variable and its negation.
- ***Simple Disjunctive Form***: A disjunctive formula composed of a finite number of literals.
  such as： p, $\neg q$, $p \vee \neg q$, $p \vee q \vee r$, …
- ***Simple Conjunctive Form***: A conjunctive formula composed of a finite number of literals.
  such as : p, $\neg q$, $p \wedge \neg q$, $p \wedge q \wedge r$, …
- Theorem 2.3:

(1) A **simple disjunctive form** is a tautology if and only if it contains both a propositional variable and its negation(e.g. $p \vee \neg q$).

(2) A **simple conjunctive form** is a contradiction if and only if it contains both a propositional variable and its negation (e.g. $p \wedge \neg q$).

- **Disjunctive Normal Form (DNF):** A disjunction composed of a finite number of simple conjunctive forms.

  $A_1 \vee A_2 \vee \ldots \vee A_r$, **where** $A_1, A_2, \ldots, A_r$ are simple conjunctive forms.

e.g. **Examples:** $p \vee q \vee \neg r$

$\neg p \wedge \neg q \wedge \neg r$

$(p_1 \wedge \neg p_2 \wedge p_3) \vee (\neg p_1 \wedge p_2 \wedge p_3) \vee (p_1 \wedge p_2 \wedge \neg p_3)$

- **Conjunctive Normal Form (CNF):** A conjunction composed of a finite number of simple disjunctive forms.

  $A_1 \wedge A_2 \wedge \ldots \wedge A_r$, **where** $A_1, A_2, \ldots, A_r$ are simple disjunctive forms.

e.g. **Examples:** $p \vee q \vee \neg r$

$\neg p \wedge \neg q \wedge r$

$(p_1 \vee \neg p_2 \vee \neg p_3) \wedge (\neg p_1 \vee p_2 \vee p_3) \wedge (p_1 \vee \neg p_3)$

- **Normal Form**: A general term referring to both disjunctive normal form (DNF) and conjunctive normal form (CNF).
- **Theorem 2.4:**

  (1) A ***disjunctive normal form*** (DNF) is a contradiction if and only if each of its simple conjunctive forms is a contradiction

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(2) A ***conjunctive normal form*** (CNF) is a tautology if and only if each of its simple disjunctive forms is a tautology.

- Theorem 2.5: ***Normal Form Existence Theorem***.

  Every propositional formula has an equivalent disjunctive normal form (DNF) and conjunctive normal form (CNF).

- **Proof:** The three important steps for obtaining the normal form of a formula *A*

  (1) Eliminate $\rightarrow$, $\leftrightarrow$ at *A*

  $$A \rightarrow B \Leftrightarrow \neg A \vee B$$

  $$A \leftrightarrow B \Leftrightarrow (\neg A \vee B) \wedge (A \vee \neg B)$$

  (2) Move negation ($\neg$) inward or eliminate it:

  $$\neg\neg A \Leftrightarrow A$$

  $$\neg(A \vee B) \Leftrightarrow \neg A \wedge \neg B$$

  $$\neg(A \wedge B) \Leftrightarrow \neg A \vee \neg B$$

(3) Using the Distributive Law

$A\vee(B\wedge C)\Leftrightarrow(A\vee B)\wedge(A\vee C)$ For finding CNF

$A\wedge(B\vee C)\Leftrightarrow(A\wedge B)\vee(A\wedge C)$ For finding DNF

e.g. **Example:** **Find** $\neg(p\rightarrow q)\vee\neg r$ DNF and CNF

Solution: $\neg(p\rightarrow q)\vee\neg r$

$\Leftrightarrow \neg(\neg p\vee q)\vee\neg r$

$\Leftrightarrow (p\wedge\neg q)\vee\neg r$ DNF

$\Leftrightarrow (p\vee\neg r)\wedge(\neg q\vee\neg r)$ CNF

Note: The DNF and CNF of a formula are not unique.

- **Definition 2.17: minterm** and **maxterm**

  In a simple conjunctive form (or simple disjunctive form) containing ***n*** propositional variables, if each propositional variable appears exactly once in the form of a literal and the ***i-th literal*** (arranged in subscript or alphabetical order) appears in the ***i-th position*** from the left, such a simple conjunctive form (or simple disjunctive form) is called a ***minterm*** (or ***maxterm***).

e.g. **Examples:** $\neg p \wedge \neg q$， $\neg p \wedge \neg q \wedge r$ are **minterms.**

$\neg p \vee \neg q$， $\neg p \vee \neg q \vee r$ are **maxterms.**

Explanation:

(1) Propositional variables serve as placeholders in propositional logic. They do not inherently have truth values but can be replaced by specific propositions with definite truth values (1 or 0).

(2) A propositional variable can either be a simple proposition or a logical combination involving other variables.

(3) A truth table with n propositional variables, ranging from all "0" to all "1," consists of 2^n rows, corresponding to 2^n minterms and 2^n maxterms.

Explanation:

(4) The 2^n minterms (or maxterms) are all distinct from each other in terms of logical equivalence.

(5) Let m_i denote the ***i*-th minterm**, where ***i*** is the decimal representation of the truth assignment that makes the minterm true. Let M_i denote the ***i*-th maxterm**, where ***i*** is the decimal representation of the truth assignment that makes the maxterm false. m_i(or M_i) is called the name of the minterm (or maxterm).

Minterms and Maxterms Formed by *p*,*q*

Minterm			Maxterm		
Formula	Ture	Name	Formula	False	Name
$\neg p \wedge \neg q$	0 0	m_0	$p \vee q$	0 0	M_0
$\neg p \wedge q$	0 1	m_1	$p \vee \neg q$	0 1	M_1
$p \wedge \neg q$	1 0	m_2	$\neg p \vee q$	1 0	M_2
$p \wedge q$	1 1	m_3	$\neg p \vee \neg q$	1 1	M_3

- **Theorem 2.6: Let** m_i M_i be the minterm and maxterm formed by the same set of propositional variables. Then:

$$\neg m_i \Leftrightarrow M_i, \quad \neg M_i \Leftrightarrow m_i$$

- ***Principal Disjunctive Normal Form*** **(PDNF):** A disjunctive normal form composed of minterms.
- ***Principal Conjunctive Normal Form*** **(PCNF):** A conjunctive normal form composed of maxterms.

e.g. **Example：** n=3, propositional variables p, q, r,

$(\neg p\wedge\neg q\wedge r)\vee(\neg p\wedge q\wedge r) \Leftrightarrow m_1\vee m_3$ (PDNF)

$(p\vee q\vee\neg r)\wedge(\neg p\vee q\vee\neg r) \Leftrightarrow M_1\wedge M_5$ (PCNF)

- **Theorem 2.7:** Every propositional formula has an equivalent PDNF and PCNF, and these forms are **unique**.

- Let formula A contain propositional variables $p_1,p_2,\ldots,p_n$, find the principal disjunctive normal form (**PDNF**) of A.

(1) **Find a Disjunctive Normal Form (DNF):** $A'=B_1\vee B_2\vee \ldots \vee B_s$, where each B_j is a **simple conjunction** , $j=1,2, \ldots ,s$.

(2) If a term B_j lacks either p_i, or $\neg p_i$, B_j expand it using

$$B_j \Leftrightarrow B_j\wedge(p_i\vee\neg p_i) \Leftrightarrow (B_j\wedge p_i)\vee(B_j\wedge\neg p_i)$$

Repeat until every conjunction becomes a **minterm** (length n).

(3) **Remove Duplicate Minterms:** Replace repeated minterms $m_i\vee m_i$ with m_i.

(4) **Order Minterms:** Arrange the minterms in ascending order of their indices.

- Let formula A contain propositional variables $p_1,p_2,\ldots,p_n$, find the principal **Conjunctive** normal form (**PCNF**) of A.

(1) **Find a Conjunctive Normal Form (CNF)** $A'=B_1\wedge B_2\wedge \ldots \wedge B_s$, where each B_j is a **simple disjunction**, $j=1,2, \ldots ,s$.

(2) If a term B_j lacks either p_i or $\neg p_i$ for some variable p_i, expand it using: $B_j \Leftrightarrow B_j\vee(p_i\wedge\neg p_i) \Leftrightarrow (B_j\vee p_i)\wedge(B_j\vee\neg p_i)$

Repeat until every disjunction becomes a **maxterm** (length n).

(3) Replace repeated maxterms (e.g., $M_i \wedge M_i$) with a single M_i.

(4) Arrange the maxterms in ascending order of their indices (e.g., M_0, M_1, M_2, …).

e.g. Example: Find the PDNF and PCNF of $\neg(p\rightarrow q) \vee \neg r$

Step 1: Find PDNF (Minterms)

$\neg(p\rightarrow q)\vee\neg r \Leftrightarrow (p\wedge\neg q)\vee\neg r$ (Implication equivalence, De Morgan)

$p\wedge\neg q \Leftrightarrow (p\wedge\neg q)\wedge 1$

$\Leftrightarrow (p\wedge\neg q)\wedge(\neg r\vee r)$

$\Leftrightarrow (p\wedge\neg q\wedge\neg r)\vee(p\wedge\neg q\wedge r)$

$\Leftrightarrow m_4\vee m_5$

$\neg r \Leftrightarrow (\neg p\vee p)\wedge(\neg q\vee q)\wedge\neg r$

$\Leftrightarrow (\neg p\wedge\neg q\wedge\neg r)\vee(\neg p\wedge q\wedge\neg r)\vee(p\wedge\neg q\wedge\neg r)\vee(p\wedge q\wedge\neg r)$

$\Leftrightarrow m_0\vee m_2\vee m_4\vee m_6$

$\neg(p\rightarrow q)\vee\neg r \Leftrightarrow m_0\vee m_2\vee m_4 \vee m_5 \vee m_6$

$\Leftrightarrow \Sigma(0,2,4,5,6)$

- Step 2: Find PCNF (Maxterms)

$$\neg(p\to q)\vee\neg r \Leftrightarrow (p\vee\neg r)\wedge(\neg q\vee\neg r)$$

$$p\vee\neg r \Leftrightarrow p\vee 0\vee\neg r$$

$$\Leftrightarrow p\vee(q\wedge\neg q)\vee\neg r$$

$$\Leftrightarrow (p\vee q\vee\neg r)\wedge(p\vee\neg q\vee\neg r)$$

$$\Leftrightarrow M_1\wedge M_3$$

$$\neg q\vee\neg r \Leftrightarrow (p\wedge\neg p)\vee\neg q\vee\neg r$$

$$\Leftrightarrow (p\vee\neg q\vee\neg r)\wedge(\neg p\vee\neg q\vee\neg r)$$

$$\Leftrightarrow M_3\wedge M_7$$

$$\neg(p\to q)\vee\neg r \Leftrightarrow M_1\wedge M_3\wedge M_7$$

$$\Leftrightarrow \Pi(1,3,7)$$

- A ***quick way*** to obtain the PDNF (or PCNF) of a formula A with n propositional variables is to complete the missing variables to form all possible minterms (or maxterms).
- A **conjunctive clause** (simple AND term) of length k can be expanded into 2^{n-k} minterms.

Such as: Formulation p,q,r

$$q \Leftrightarrow (\neg p\vee q\vee\neg r)\wedge(\neg p\vee q\vee r)\wedge(p\vee q\vee\neg r)\wedge(p\vee q\vee r)$$

$$\Leftrightarrow m_2\vee m_3\vee m_6\vee m_7$$

- **A disjunctive clause** (simple OR term) of length k can be expanded into 2^{n-k} maxterms.
- **Such as:** $p\vee\neg r \Leftrightarrow (p\vee q\vee\neg r)\wedge(p\vee\neg q\vee\neg r)$

$$\Leftrightarrow M_1\wedge M_3$$

e.g. **Example:** Find the PDNF (Principal Disjunctive Normal Form)

$$A \Leftrightarrow (\neg p \wedge q) \vee (\neg p \wedge \neg q \wedge r) \vee r$$

Solution (Quick Method):

$$\neg p \wedge q \Leftrightarrow (\neg p \wedge q \wedge \neg r) \vee (\neg p \wedge q \wedge r) \Leftrightarrow m_2 \vee m_3$$

$$\neg p \wedge \neg q \wedge r \Leftrightarrow m_1$$

$$r \Leftrightarrow (\neg p \wedge \neg q \wedge r) \vee (\neg p \wedge q \wedge r) \vee (p \wedge \neg q \wedge r) \vee (p \wedge q \wedge r)$$

$$\Leftrightarrow m_1 \vee m_3 \vee m_5 \vee m_7$$

$$A \Leftrightarrow m_1 \vee m_2 \vee m_3 \vee m_5 \vee m_7 \Leftrightarrow \Sigma(1,2,3,5,7)$$

e.g. **Example:** Find the PCNF (Principal Conjunctive Normal Form)

$$B \Leftrightarrow \neg p \wedge (p \vee q \vee \neg r)$$

■ **Solution (Quick Method):**

$$\neg p \Leftrightarrow (\neg p \vee q \vee r) \wedge (\neg p \vee q \vee \neg r) \wedge (\neg p \vee \neg q \vee r) \wedge (\neg p \vee \neg q \vee \neg r)$$

$$\Leftrightarrow M_4 \wedge M_5 \wedge M_6 \wedge M_7$$

$$p \vee q \vee \neg r \Leftrightarrow M_1$$

Get $B \Leftrightarrow M_1 \wedge M_4 \wedge M_5 \wedge M_6 \wedge M_7 \Leftrightarrow \Pi(1,4,5,6,7)$

- Use PDNF to Determine the ***True Assignments*** and ***False Assignments*** of a Formula.
- Let formula *A* contain ***n*** propositional variables. If the PDNF (Principal Disjunctive Normal Form) of *A* has ***s*** minterms, then:
  - ***True Assignments***: ***s*** assignments corresponding to the binary representations of the minterm indices.
  - ***False Assignments***: 2^{n}-s assignments not covered by the minterms.

e.g. **Example:** $\neg(p\rightarrow q)\vee\neg r \Leftrightarrow m_0\vee m_2\vee m_4\vee m_5\vee m_6$

True Assignments: 000,010,100,101,110;

False Assignments: 001,011,111

- Use PDNF to Determine the type of a Formula.
- Let *A* be a formula with ***n*** propositional variables.
  - *A* is a ***tautology* if and only if** its PDNF (Principal Disjunctive Normal Form) contains **all 2^n minterms.**
  - *A* is a ***contradiction* if and only if** its PDNF contains **no minterms** (denoted as 0).
  - *A* is ***satisfiable* if and only if** its PDNF contains **at least one minterm.**

e.g. **Example:** Determining formula types using principal disjunctive normal form (PDNF).

- **Problem:** Classify the following formulas:

  (1) $A \Leftrightarrow \neg(p \rightarrow q) \wedge q$　(2) $B \Leftrightarrow p \rightarrow (p \vee q)$　(3) $C \Leftrightarrow (p \vee q) \rightarrow r$

- **Solution:**

  (1) $A \Leftrightarrow \neg(\neg p \vee q) \wedge q \Leftrightarrow (p \wedge \neg q) \wedge q \Leftrightarrow 0$　*contradiction*

  (2) $B \Leftrightarrow \neg p \vee (p \vee q) \Leftrightarrow 1 \Leftrightarrow m_0 \vee m_1 \vee m_2 \vee m_3$　*tautology*

  (3) $C \Leftrightarrow \neg(p \vee q) \vee r \Leftrightarrow (\neg p \wedge \neg q) \vee r$

  $\Leftrightarrow (\neg p \wedge \neg q \wedge r) \vee (\neg p \wedge \neg q \wedge \neg r) \vee (\neg p \wedge \neg q \wedge r)$

  $\vee (\neg p \wedge q \wedge r) \vee (p \wedge \neg q \wedge r) \vee (p \wedge q \wedge r)$

  $\Leftrightarrow m_0 \vee m_1 \vee m_3 \vee m_5 \vee m_7$　*Non-tautological satisfiable formula*

- Use PDNF to determine whether two formulas are logically equivalent.

e.g. **Example:** Use principal disjunctive normal form (PDNF) to determine if the following pair of formulas are equivalent:

(1) p and $(\neg p \vee q) \rightarrow (p \wedge q)$

Solve: $p \Leftrightarrow p \wedge (\neg q \vee q) \Leftrightarrow (p \wedge \neg q) \vee (p \wedge q) \Leftrightarrow m_2 \vee m_3$

$(\neg p \vee q) \rightarrow (p \wedge q) \Leftrightarrow \neg(\neg p \vee q) \vee (p \wedge q)$

$\Leftrightarrow (p \wedge \neg q) \vee (p \wedge q) \Leftrightarrow m_2 \vee m_3$

Then: $p \Leftrightarrow (\neg p \vee q) \rightarrow (p \wedge q)$

e.g. **Example:** Use principal disjunctive normal form (PDNF) to determine if the following pair of formulas are equivalent:

(2) $(p\wedge q)\vee r$ and $p\wedge(q\vee r)$

Solve: $(p\wedge q)\vee r \Leftrightarrow (p\wedge q\wedge\neg r)\vee(p\wedge q\wedge r)$

$\vee(\neg p\wedge\neg q\wedge r)\vee(\neg p\wedge q\wedge r)\vee(p\wedge\neg q\wedge r)\vee(p\wedge q\wedge r)$

$\Leftrightarrow m_1\vee m_3\vee m_5\vee m_6\vee m_7$

$p\wedge(q\vee r) \Leftrightarrow (p\wedge q)\vee(p\wedge r)$

$\Leftrightarrow(p\wedge q\wedge\neg r)\vee(p\wedge q\wedge r)\vee(p\wedge\neg q\wedge r)\vee(p\wedge q\wedge r)$

$\Leftrightarrow m_5\vee m_6\vee m_7$

Then: $(p\wedge q)\vee r$ is not equal to $p\wedge(q\vee r)$

e.g. **Example:** A company needs to select personnel from A, B, and C for an overseas assignment, subject to the following conditions: (1) If A goes, then C must go; (2) If B goes, then C cannot go; (3) Exactly one of A or B must go.

- **Question:** How many possible selection plans are there?
- **Solution:**
  - Define propositions: p: Send A ; q: Send B; r: Send C

    (1) $p \to r$, (2) $q \to \neg r$, (3) $(p \wedge \neg q) \vee (\neg p \wedge q)$
  - Combine into a logical formula:

$$C = (p \to r) \wedge (q \to \neg r) \wedge ((p \wedge \neg q) \vee (\neg p \wedge q))$$

↳ Use PDNF to designate the personnel selection plan(cont.)

- Find principal disjunctive normal form of *C*

$$C=(p\to r)\wedge(q\to\neg r)\wedge((p\wedge\neg q)\vee(\neg p\wedge q))$$

$$\Leftrightarrow (\neg p\vee r)\wedge(\neg q\vee\neg r)\wedge((p\wedge\neg q)\vee(\neg p\wedge q))$$

$$\Leftrightarrow ((\neg p\wedge\neg q)\vee(\neg p\wedge\neg r)\vee(r\wedge\neg q)\vee(r\wedge\neg r))$$

$$\wedge((p\wedge\neg q)\vee(\neg p\wedge q))$$

$$\Leftrightarrow ((\neg p\wedge\neg q)\wedge(p\wedge\neg q))\vee((\neg p\wedge\neg r)\wedge(p\wedge\neg q))$$

$$\vee((r\wedge\neg q)\wedge(p\wedge\neg q))\vee((\neg p\wedge\neg q)\wedge(\neg p\wedge q))$$

$$\vee((\neg p\wedge\neg r)\wedge(\neg p\wedge q))\vee((r\wedge\neg q)\wedge(\neg p\wedge q))$$

$$\Leftrightarrow (p\wedge\neg q\wedge r)\vee(\neg p\wedge q\wedge\neg r)$$

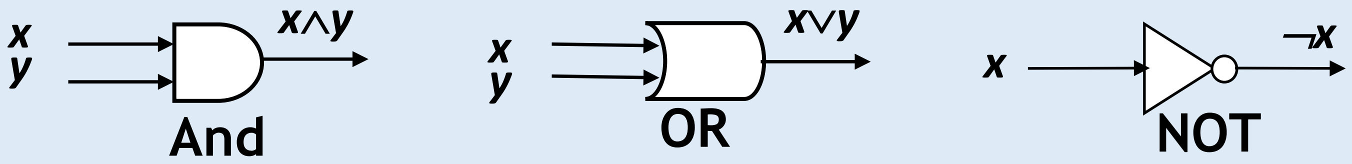
- **True Assignment :101,010**

Plan 1: Send A and C

Plan 2: Send B

↳ Use PDNF to design a control circuit

e.g. **Example:** A lamp is controlled by two switches. Pressing either switch can turn the lamp on or off. Use the gate circuits in the diagram to design the control circuit.



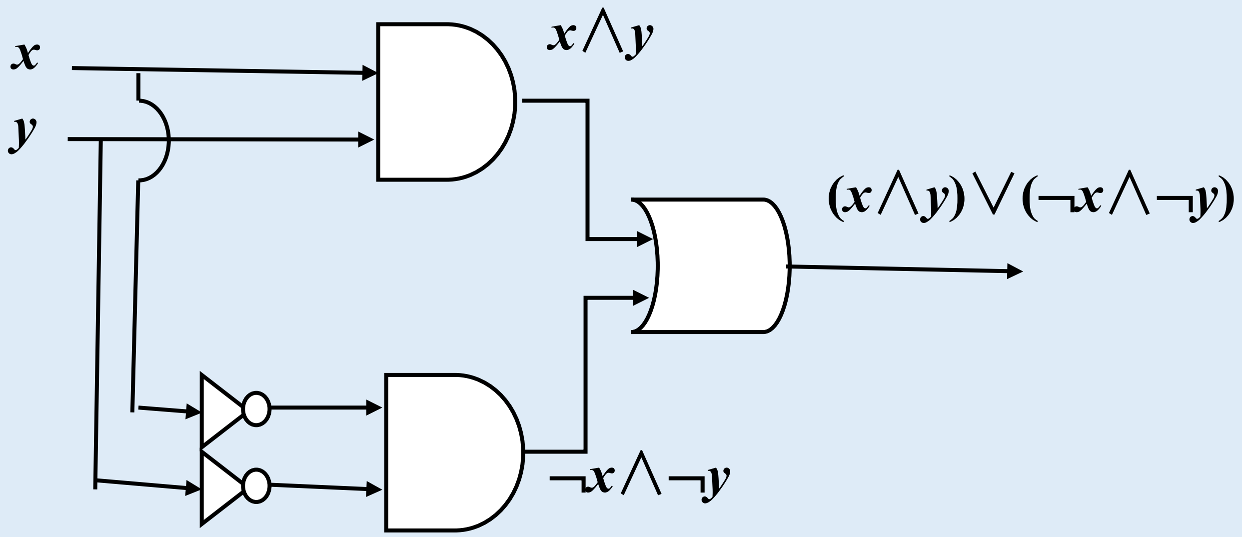
■**Solution:**

- Let ***x*** and ***y*** represent the states of the two switches (0 = off, 1 = on).
- Let *F* represent the lamp's state (1 = on, 0 = off).
- Assume the lamp is initially on (F=1) when x=y=0.
- **Canonical DNF** (Principal Disjunctive Normal Form) **of *F***

$$F=m_0\wedge m_3$$
$$= (\neg x\wedge\neg y)\vee(x\wedge y)$$

x	*y*	*F*
0	0	1
0	1	0
1	0	0
1	1	1

↳ Use PDNF to design a control circuit (cont.)




If the initial condition is set such that the light turns on (***F***=1) only when exactly one of ***x*** or ***y*** is '1', how would the circuit behave?

↳ Convert PDNF to PCNF

Let: $A \Leftrightarrow m_{i_1} \vee m_{i_2} \vee \cdots \vee m_{i_s}$

Non exited Minterm is $m_{j_1}, m_{j_2}, \cdots, m_{j_t}$, So $t=2^{n-s}$,

Thus:

$$\neg A \Leftrightarrow m_{j_1} \vee m_{j_2} \vee \cdots \vee m_{j_t}$$

$$A \Leftrightarrow \neg(m_{j_1} \vee m_{j_2} \vee \cdots \vee m_{j_t})$$

$$\Leftrightarrow \neg m_{j_1} \wedge \neg m_{j_2} \wedge \cdots \wedge \neg m_{j_t}$$

$$\Leftrightarrow M_{j_1} \wedge M_{j_2} \wedge \cdots \wedge M_{j_t}$$

Objective：

Key Concepts：